QUANTITATIVE FINANCE MADE ACCESSIBLE

COURSE N°3:

« Course 3: Poisson Processes and Jump Diffusion »

Saturday, October 14, 2023

Florian CAMPUZAN, CFA

FINANCE TUTORING

Conseil et Formation

AGENDA FOR THE SESSION

❖ **Course 3: Poisson Processes and Jump Diffusion**

1. Incorporating Poisson processes to model rare events and jumps in prices
2. Applying jump diffusion models to capture extreme market movements
3. Mini Case Study: An insurance company models the occurrence of large-scale catastrophic events using a Poisson process to estimate potential claims.

I-WHAT ARE POISSON PROCESSES?

UNDERSTANDING THE POISSON PROCESSES

Understanding the poisson processes

A Poisson process is a type of **stochastic process** where events happen randomly in continuous time. These events are independent of each other, and they happen at a constant average rate.

A **stochastic process** describes the evolution of a system over time in the presence of uncertainty.

Examples:

Random Walk: This is a simple stochastic process where the state at each time step is the previous state plus some random change.

Markov Chains: This is a type of stochastic process where the probability of being in a particular state at time tt depends only on the state at time $t - 1$.

Brownian Motion: Also known as Wiener process, it's a continuous-time stochastic process where the changes in the state from one time to the next are normally distributed.

THE GEOMETRIC BROWNIAN MOTION

- **The differential equation representing GBM is:**

- $dSt = \mu Stdt + \sigma StdWt$

where,

✓ St: the stock price at time t,

✓ μ: the expected return or "drift" coefficient,

✓ σ: the volatility or "diffusion" coefficient,

✓ Wt: a standard Brownian motion.

UNDERSTANDING THE POISSON PROCESSES

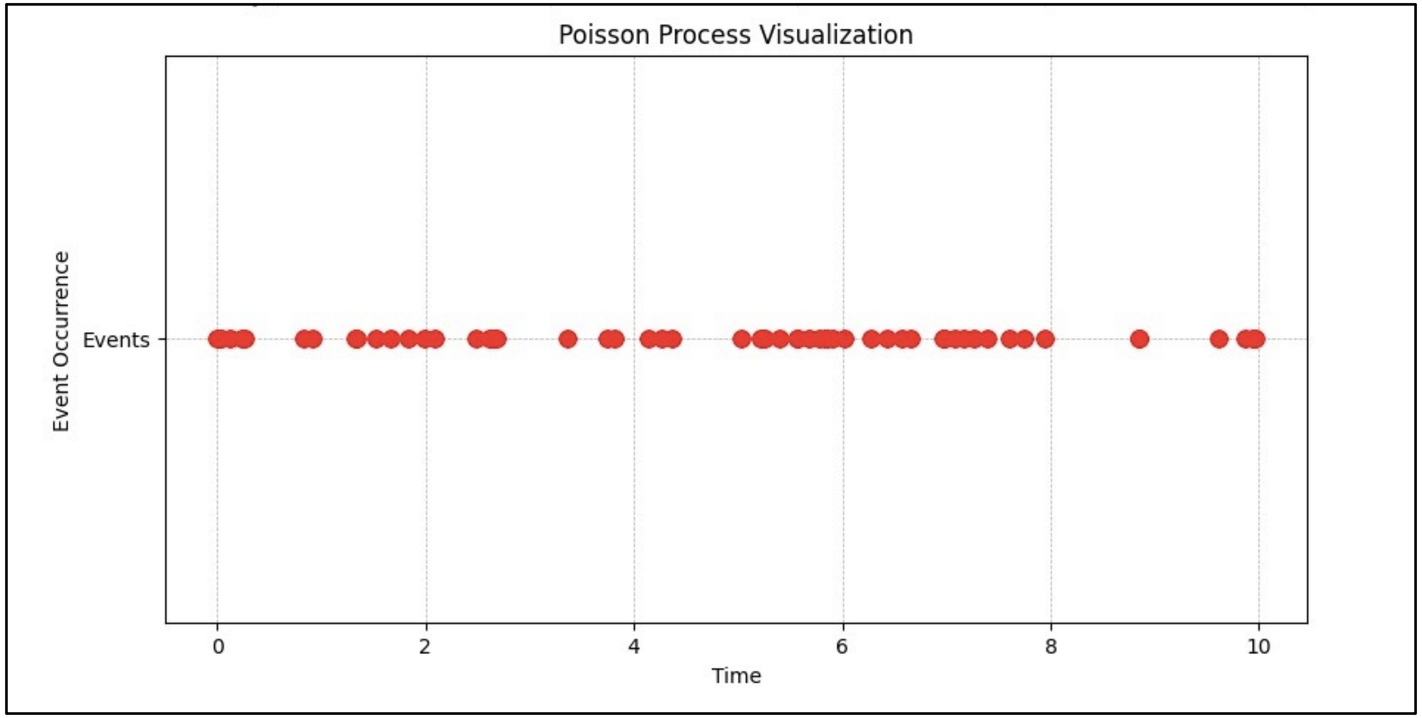
The Poisson process is a stochastic process that represents the number of events occurring in fixed intervals of time or space. These events happen with a known constant mean rate and are independent of the time since the last event.

Here's how to interpret the points: Each red dot represents an event that has occurred in continuous time.

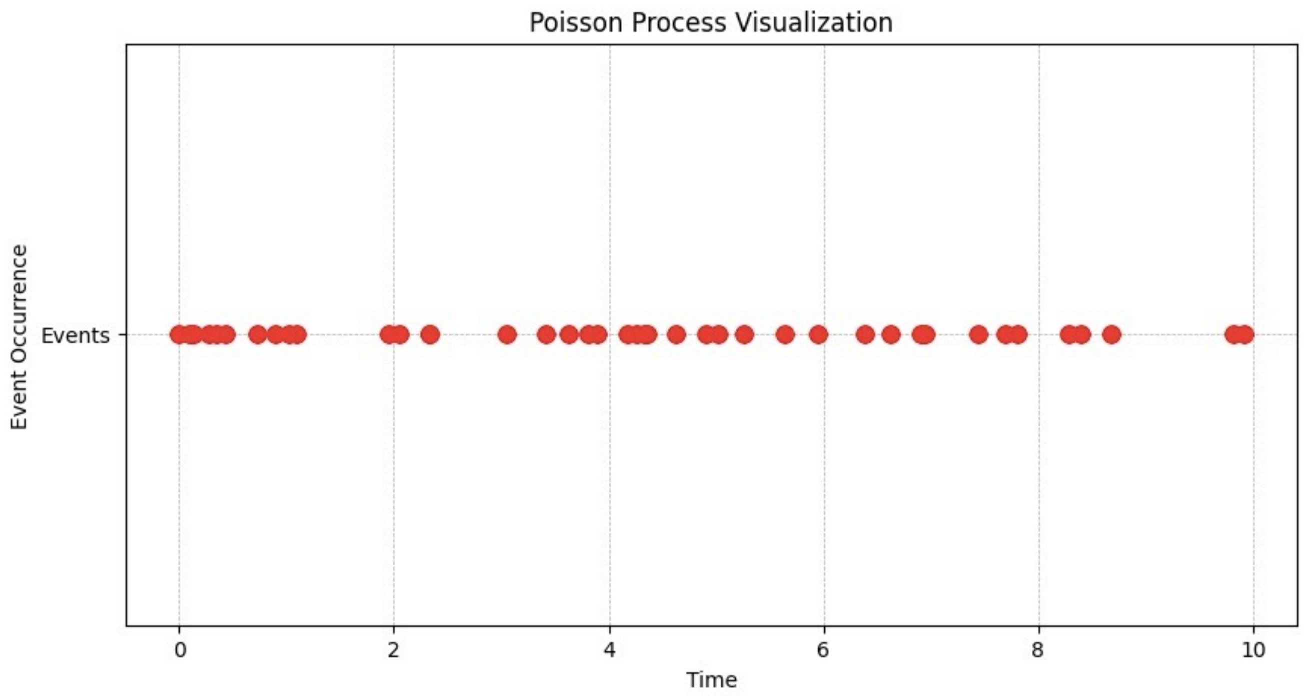
x-coordinate (Time): The horizontal position of the dot corresponds to the exact time when an event occurred. So, if a dot is located at x = 3.2, this means an event occurred at time t = 3.2 units.

y-coordinate (Event Occurrence): In our script, this is set to a constant value of 1. It does not indicate the magnitude or intensity of the event. Instead, its purpose is merely to depict the event's occurrence visually.

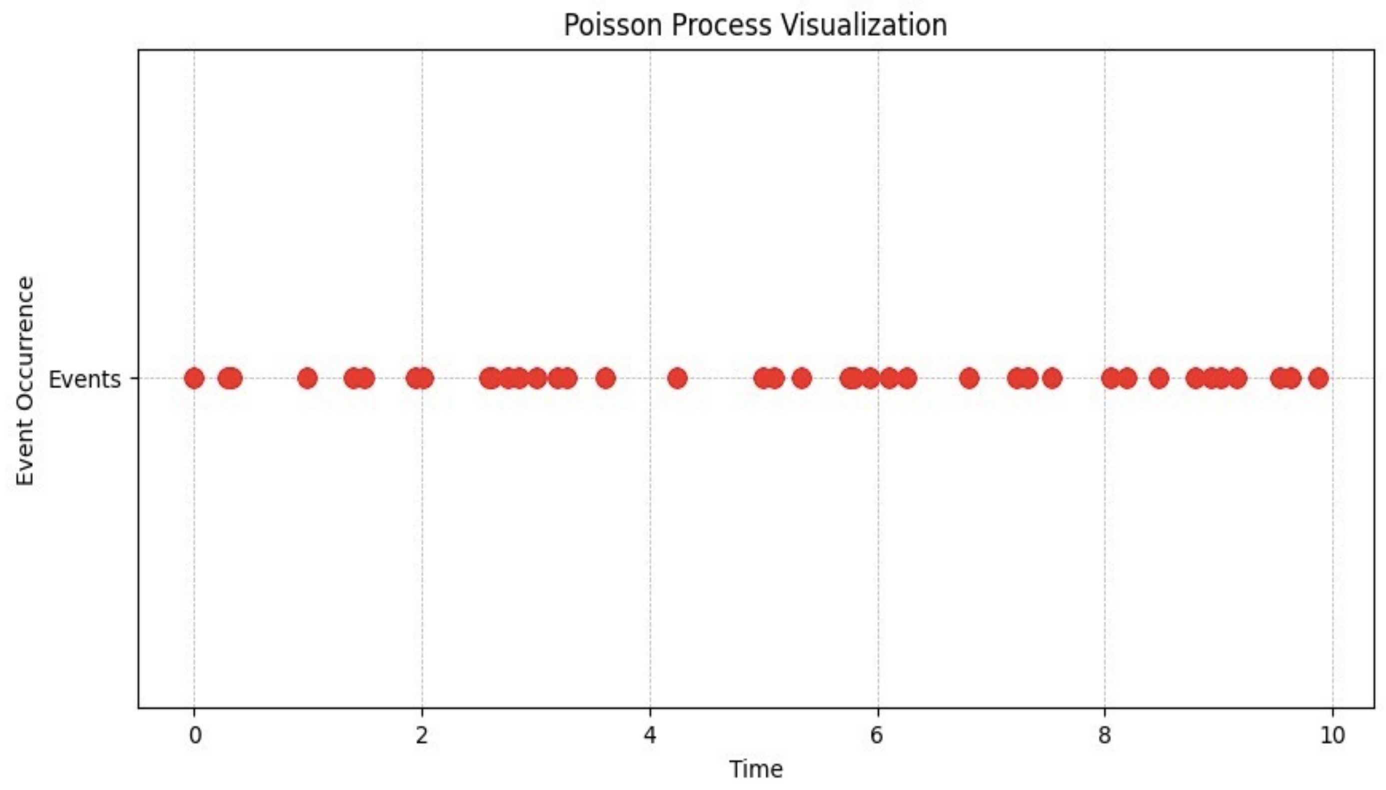
UNDERSTANDING THE POISSON PROCESSES



UNDERSTANDING THE POISSON PROCESSES



UNDERSTANDING THE POISSON PROCESSES



UNDERSTANDING THE POISSON PROCESSES:

THE MODEL

❖ Poisson Distribution

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

- $P(X = k)$: Probability of observing k events in a fixed interval.
- The $\mathbf{e}^{-\lambda}$ term provides the probability that zero events happen in the specified interval.
- λ (lambda) represents the average rate of occurrence of events per unit of time or space.
- The term λ^k: models the likelihood of observing exactly k events in the interval. λ gives us an average or expected rate, while k is a specific count of events for which we want to determine the probability.
- $k!$ (k factorial): k is the product of all positive integers less than or equal to k. It's used as a normalizing factor in the formula to ensure the probabilities sum up to 1.

UNDERSTANDING THE POISSON PROCESSES:

EXAMPLE

❖ **Price Jumps in the Stock Market**

- Imagine you are analyzing the stock price of a company that has been relatively stable but occasionally experiences significant price jumps due to unpredictable events like surprise earnings announcements, regulatory changes, or acquisition news.
- For our scenario, let's say you've observed that, on average, the stock experiences 5 significant price jumps per year.

UNDERSTANDING THE POISSON PROCESSES:

INTERPRETATION

❖ **Formula Components**:

- $e^{-\lambda}$:
  - ✓ This term represents the probability of no price jumps occurring in a year, given our rate. For λ=5, e^{-5} is approximately 0.0067, so there's a 0.67% chance of observing no price jumps in a year.
- λ^k:
  - ✓ This term is crucial when we want to find the probability of a specific number of price jumps occurring in a year.
  - ✓ For instance, let's consider k=3 (i.e., we want to know the probability of observing exactly 3 price jumps in a year). Here, $\lambda^k=5^3=125$.
- $k!$: This term is a normalizing factor. For our k=3 scenario, $k!=3\times2\times1=6k!=3\times2\times1=6$.
  - ✓ Plugging in the numbers: $P(X=3)$=0.0067×1256≈0.1408

II-MODELING PRICE JUMPS WITH THE JUMP DIFFUSION MODEL

BRIDGING THE POISSON DISTRIBUTION TO THE JUMP DIFFUSION MODE

The Jump Diffusion model uses the Poisson process to model the occurrence of these sudden jumps. Here's how:

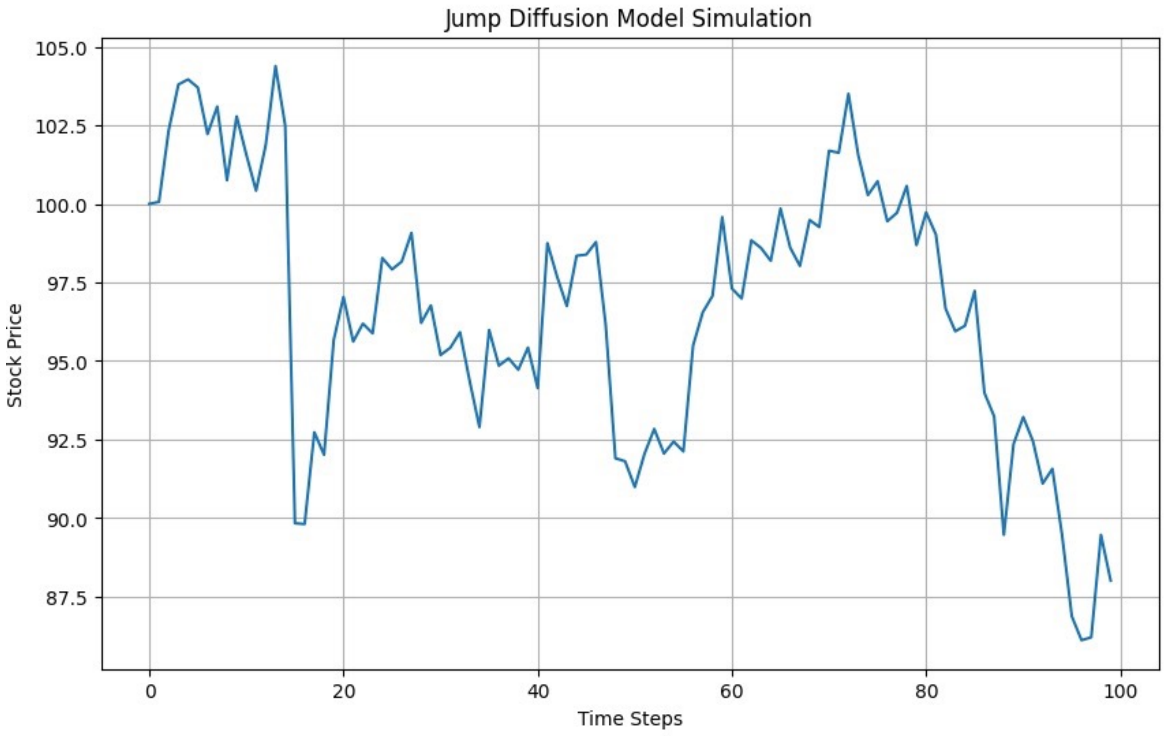
Poisson Process Nt:The jumps occur according to a Poisson process with rate λ. This means that in any small interval *dt*, there's a λdt probability of a jump occurring.

Jump Size Jt:When a jump occurs, the size of that jump is determined by some distribution, often log-normal..

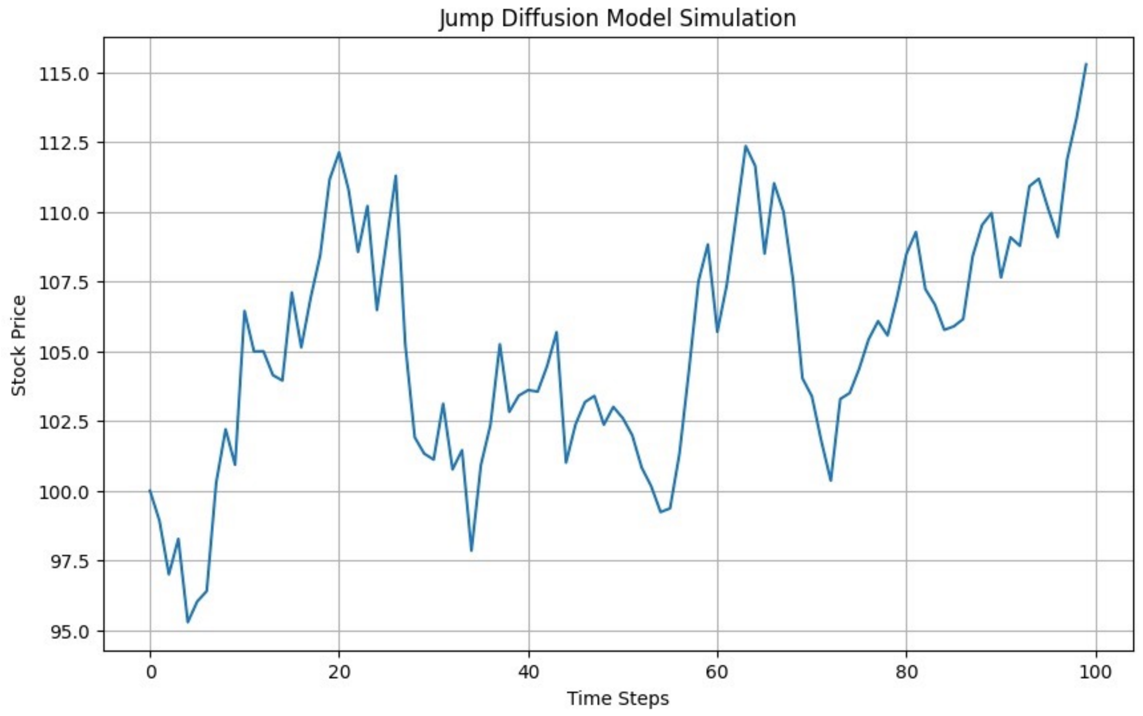
The timing of the jumps is random and follows the Poisson distribution with parameter λ. The actual magnitude of those jumps (up or down) is typically modeled with another distribution, like the normal distribution.

By combining this with the Geometric Brownian Motion, the Jump Diffusion model gives a more comprehensive picture of asset price movements.

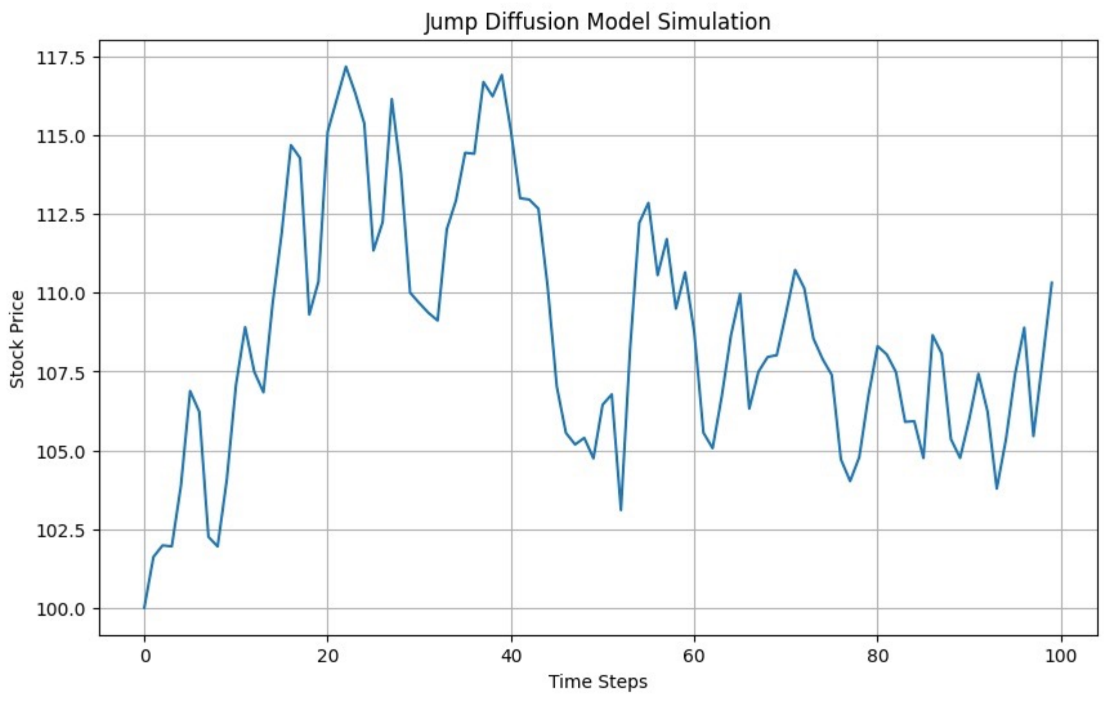
UNDERSTANDING THE JUMP DIFFUSION MODEL



UNDERSTANDING THE JUMP DIFFUSION MODEL



UNDERSTANDING THE JUMP DIFFUSION MODEL



UNDERSTANDING THE JUMP DIFFUSION MODEL:

THE MODEL

❖ The jump diffusion model

$$dS_t = \mu S_t dt + \sigma S_t dW_t + J_t dN_t$$

- ***μStdt***: expected return part, the regular drift of the stock.
- ***σStdWt***: represents the volatility, the continuous part of the stock's movement
- ***JtdNt***: represents the jump part, where ***Jt*** is the magnitude of the jump and ***dNt*** is the increment of the Poisson process (either 0 indicating no jump or 1 (indicating a jump).

➢ **In essence, the formula captures:**

1. The normal drift and volatility of a stock.
2. The sudden jumps in stock price.
3. Using this model, financial analysts can more accurately price derivatives and assess the risks associated with assets that may experience sudden, significant price changes.

UNDERSTANDING THE POISSON PROCESSES:

EXAMPLE

❖ **Price Jumps in the Stock Market**

- Imagine you are analyzing the stock price of a company that has been relatively stable but occasionally experiences significant price jumps due to unpredictable events like surprise earnings announcements, regulatory changes, or acquisition news.
- For our scenario, let's say you've observed that, on average, the stock experiences 5 significant price jumps per year.

UNDERSTANDING THE POISSON PROCESSES:

EXAMPLE

❖ **Formula Components**:

- $e^{-\lambda}$:

  ✓ This term represents the probability of no price jumps occurring in a year, given our rate. For λ=5λ=5, e^{-5} is approximately 0.0067, so there's a 0.67% chance of observing no price jumps in a year.

- λ^{k} :

  ✓ This term is crucial when we want to find the probability of a specific number of price jumps occurring in a year.

  ✓ For instance, let's consider k=3 (i.e., we want to know the probability of observing exactly 3 price jumps in a year). Here, λ^{k}=5^3=125.

- **k!**: This term is a normalizing factor. For our $k = 3$ scenario, $k! = 3\times2\times1 = 6k! = 3\times2\times1 = 6$.

  ✓ Plugging in the numbers: $P(X = 3)$=0.0067×1256≈0.1408

III-MINI CASE STUDY: MODELING CATASTROPHIC EVENTS WITH A POISSON PROCESS

INTRODUCTION

- In the insurance sector, accurately predicting and preparing for rare large-scale events, such as natural disasters, is crucial.
- Such events can lead to a large number of claims in a very short period.
- An insurance company is seeking a mathematical model to help them estimate potential claims arising from such catastrophic events.

WHY THE POISSON PROCESS?

❖ The Poisson process is well-suited to model rare events. The main reasons include:

1. Independence: The occurrence of one event doesn't affect the probability of another event happening.
2. Uniformity: Events are equally likely to happen at any time.
3. Rarity: The probability of two events happening in a very short time frame is virtually zero.

IMPLEMENTATION

- **1. Data Collection**: The company collects data over several years, noting the number of large-scale catastrophic events (earthquakes, hurricanes, floods, etc.) occurring each year.
- **2. Determine λ (Lambda)**: With this data, they determine an average rate of occurrence, λ, which represents the expected number of events per year.
- **3. Modeling Potential Claims**: Using the formula for the Poisson distribution, They can now estimate the probability of k*k* large-scale events happening in a given year.

IMPLEMENTATION

- **Results**:
  - ✓Suppose $\lambda = 2$, meaning on average, two large-scale events occur every year. The insurance company can then determine:
  - ✓The probability of no events occurring in the next year (which might be $P(K = 0)$).
  - ✓The probability of one event occurring (which might be $P(K = 1)$).
  - ✓The probability of two events occurring, and so forth.

DECISION MAKING

- If $P(K = 3)$ is found to be high, the insurance company knows there's a significant chance of three large-scale events happening in the next year. They can then adjust their financial reserves and premiums accordingly.
- The company can also develop new insurance products tailored to the needs of areas particularly susceptible to these events.

THANK YOU…