QUANTITATIVE FINANCE MADE ACCESSIBLE

COURSE N°2:

« Stochastic Processes in Finance»

Saturday, October 7, 2023

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FINANCE TUTORING

Conseil et Formation

AGENDA

- Hello everyone, thank you for your interest in this webinar entitled "Quantitative Finance Made Accessible." This second part of a series of 7 sessions will enable us to discover the world of stochastic calculus trough randow walk, ABM and GBM leading to the famous BS model
- First, let's take a quick look at the program for this session.

AGENDA

📚 **Course Program: Our GUEST SPEAKER Brian LO**

1. Brian has enriched the financial sector for over two decades, holding pivotal roles at DBS Bank, BNP Paribas, and Citibank. At DBS, as Managing Director, he spearheaded innovations in market and liquidity risk frameworks and introduced AI/ML into Asset and Liability Management (ALM). His educational feats include a PhD in Mathematics from Penn State University and an MS in Pure Math from Brown University 🎓.
2. The webinar will begin with his 15-minute talk on option pricing, followed by a 15-minute Q&A session.
3. The following half-hour will be devoted to a theoretical review of the fundamentals of stochastic calculus, from Brownian motion (arithmetic and geometric) to the Black & Scholes model. The approach will be intuitive and I won't go into mathematical demonstrations.
4. Support material will be sent to participants along with a recording of the session after the webinar. **The event will last for one hour.**

I-INTRODUCTION

UNDERSTANDING STOCHASTIC PROCESSES

- **Understanding Stochastic Processes:**

- **Definition:**

- Stochastic Process: it's a sequence of random variables representing the evolution of a system over time in an uncertain environment.
- Highlight unpredictability, the role of probability, and time dependence.

- **Real-World Analogies:**

- Weather Forecasting: despite having data and models, there's always an element of uncertainty.
- Stock Prices: Describe the unpredictable nature of stock prices and how they can be modeled as a stochastic process.

II-THE RANDOM WALK

II-THE RANDOM WALK

❖ THE SYMETRIC RANDOM WALK

- *A random walk is a mathematical concept and statistical model that describes a path consisting of a succession of random steps*. It is often used in various fields such as physics, chemistry, economics, and computer science, among others.
- In a random walk, an object or variable takes steps randomly, and each step is independent of the previous one.
- In financial markets, the random walk hypothesis is used to model the behavior of asset prices, suggesting that they follow a random walk and thus cannot be predicted.
- In a simple random walk, the magnitude of each step is typically fixed but the direction changes randomly.
- However, variations of the random walk model can include steps of variable size. For instance, in a "Lévy flight", the step sizes are drawn from a probability distribution with a heavy tail, meaning that occasionally the walker can take a very long step.

II-THE RANDOM WALK

❖THE SYMETRIC RANDOM WALK

➢A symmetric random walk is a special case of a random walk where the probabilities of moving in different directions are equal. It possesses the property of symmetry because of the equal probabilities associated with each step. In the context of a one-dimensional random walk, this means that at each step, there is a 50% chance of moving forward and a 50% chance of moving backward.

- **Mathematical Representation:**
- For a simple symmetric one-dimensional random walk:

✓ $X(n+1)\ \ with\ probability\ 0.50,$

✓ $X(n-1)\ with\ probability\ 0.50,$

- Where Xn is the position of the walker at the n th step,$X(n+1)$is the position of the walker at the $(n+1)th\ step.$

II-THE RANDOM WALK

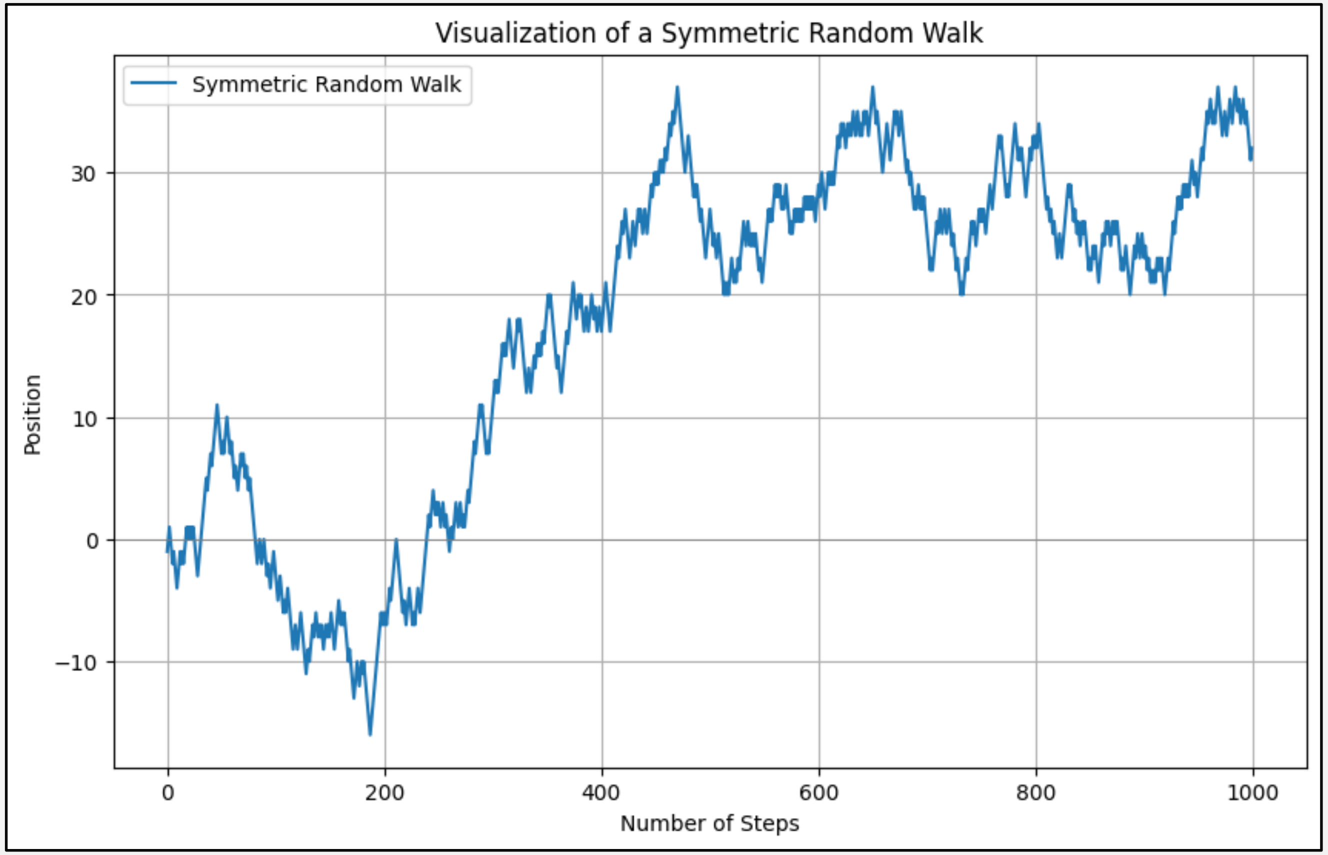
THE SYMETRIC RANDWALK

Properties:

- **Mean:** The expected position after n steps is 0 because of symmetry.
- **Variance:** The variance increases linearly with the number of steps, specifically it's n for a one-dimensional random walk.
- **Stationary Increments:** The increments are stationary, meaning the probability distribution of the walker's position doesn't change over time.
- **Markov property** (« Memorylessness ») **:** Each step is independent of the previous steps $\boldsymbol{P(Xn+1 = x \mid Xn = xn, Xn\text{-}1 = xn\text{-}1, \dots, X0 = x0) = P(Xn+1 = x \mid Xn = xn)}$
- **Martingale property** (« Fair game »)

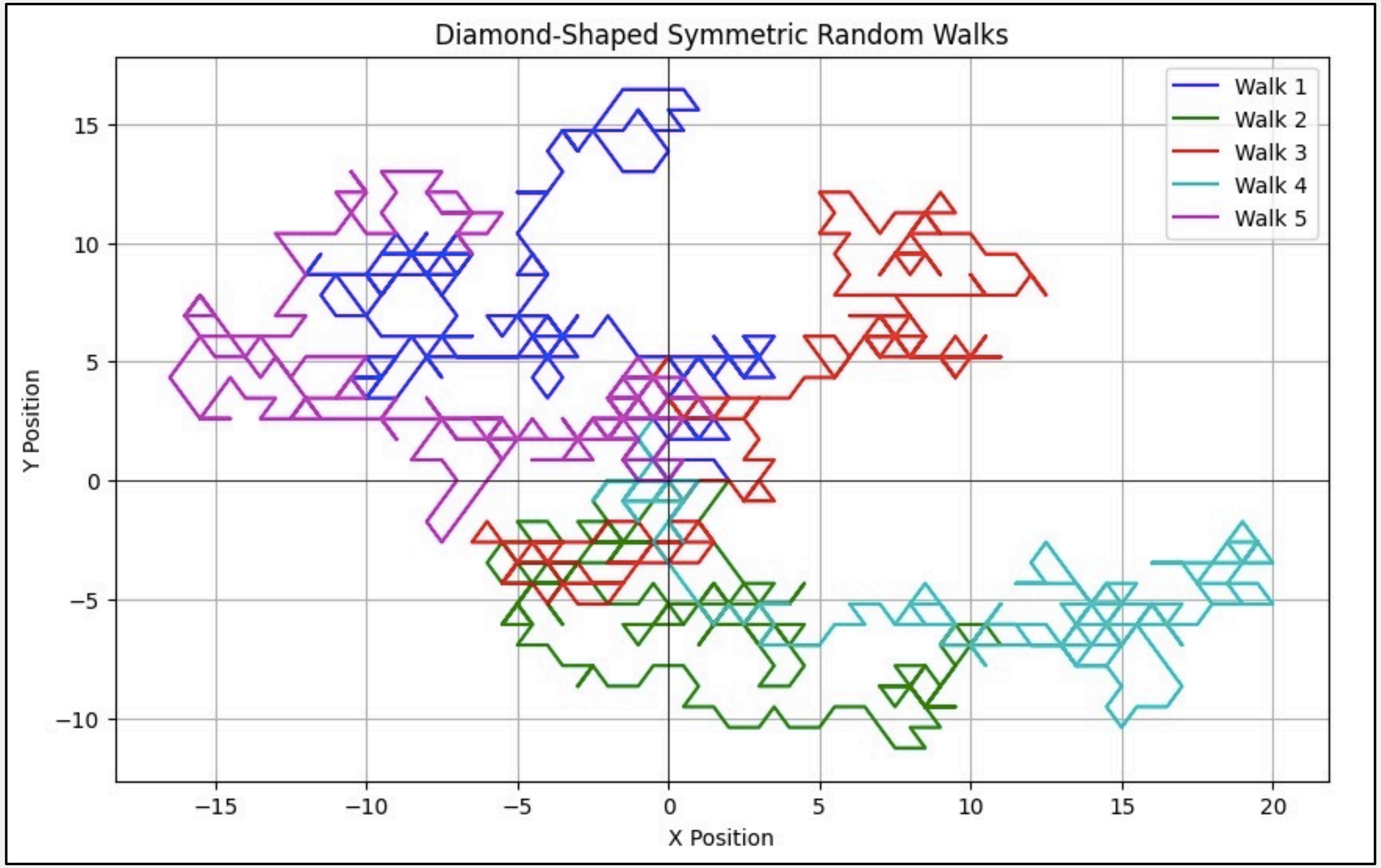
✓ $\boldsymbol{E[Xt+1 \mid X1, X2, \dots, Xt] = Xt}$ for all t. This means that the expected value of the process at the next step, given all the past information, is equal to the current value.

II-THE RANDOM WALK



- **Exploring the Arithmetic Brownian Motion**
- **Random Walk vs. Brownian Motion:**
- A symmetric random walk in continuous time approaches a Brownian motion as the time interval between steps tends to zero, and the size of steps is adjusted accordingly.
- In this context, Brownian motion can be considered as the limit of a symmetric random walk as the step size tends to zero.
- In this example, we are assuming a one-dimensional random walk, where at each step, the walker either moves forward or backward with equal probability.

II-THE RANDOM WALK



III-THE ARITHMETIC BROWNIAN MOTION

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❖ Exploring the Arithmetic Brownian Motion

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III-THE ARITHMETIC BROWNIAN MOTION

Definition of the Arithmetic Brownian motion

- **Arithmetic Brownian Motion (ABM)** is a type of stochastic process that represents a random walk in continuous time.
- It's often used in mathematical finance and physics to model random behavior over time, though it's worth noting that in financial modeling, ABM is a simpler model and **isn't used to model stock prices** as it allows for negative values.
- **Geometric Brownian Motion (GBM)** is more commonly used for this purpose.

- The term "linear" in the context of Arithmetic Brownian Motion (ABM) refers to the fact that the change in the process over time is proportional to the time increment, and the increments of the process itself are additive. This is in contrast to processes like Geometric Brownian Motion, where the process is multiplicative and exponential in nature.

III-THE ARITHMETIC BROWNIAN MOTION

❖ **Mathematical representation the Arithmetic Brownian motion**

$$dSt = \mu dt + \sigma dWt$$

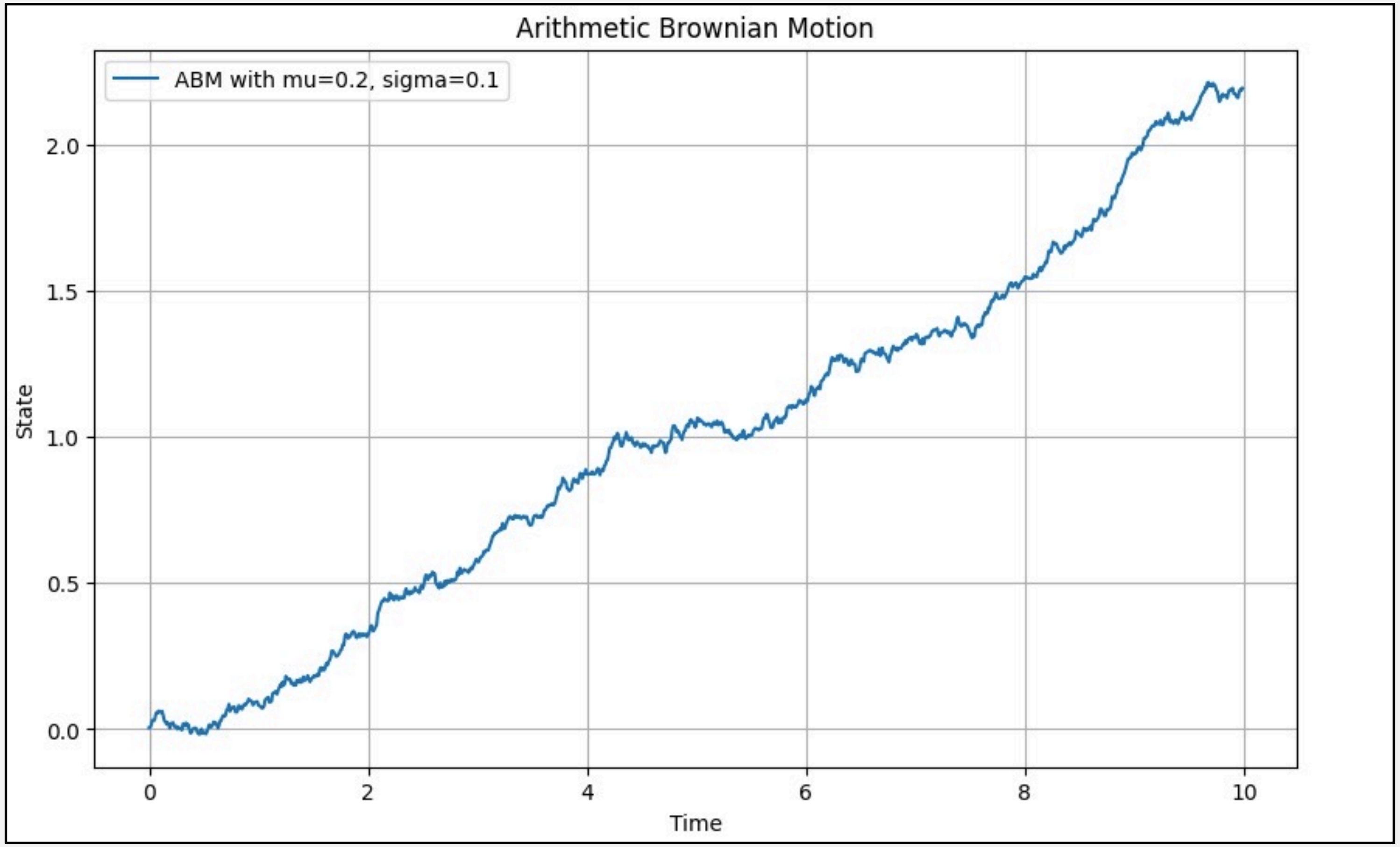
- $\boldsymbol{dSt}$: The change in the price of a financial security or asset (like a stock) over an infinitesimally small time interval.
- $\boldsymbol{\mu}$: The expected return or drift term, which represents the average expected return of the security.
- $\boldsymbol{dt}$: An infinitesimally small increment of time.
- **σ:** The volatility term, which represents how much the security's returns can vary.
- $\boldsymbol{dWt}$: The Wiener process or Brownian motion term, which introduces randomness or uncertainty into the model.
  - ✓ **It's a random variable that has a mean of zero and variance of *dt*.**

III-THE ARITHMETIC BROWNIAN MOTION

- **Exploring the Arithmetic Brownian Motion**

The chart of the Arithmetic Brownian Motion (ABM) is upward sloping primarily due to the drift term μ, which in the provided code is set to 0.2. The drift term represents a constant trend or direction in the process over time. When μ>o it introduces a positive or upward trend to the ABM.

III-THE ARITHMETIC BROWNIAN MOTION



III-THE ARITHMETIC BROWNIAN MOTION

Characteristics of Brownian Motion:

1. Continuous Paths:

- Brownian motion has continuous paths, meaning that the particle's position changes continuously over time. **This contrasts with the discrete steps in a simple random walk.**

2. Gaussian Increments:

The changes in position over fixed time intervals follow a normal (Gaussian) distribution. The increments are independent of each other.

$$\boldsymbol{B(t + \Delta t) - B(t) \sim N(0, \Delta t)}$$

This formalizes the idea that over small time intervals, the particle's position changes by an amount that is approximately normally distributed.

III-THE ARITHMETIC BROWNIAN MOTION

❖ **Characteristics of Brownian Motion:**

3. Stationary and Independent Increments:

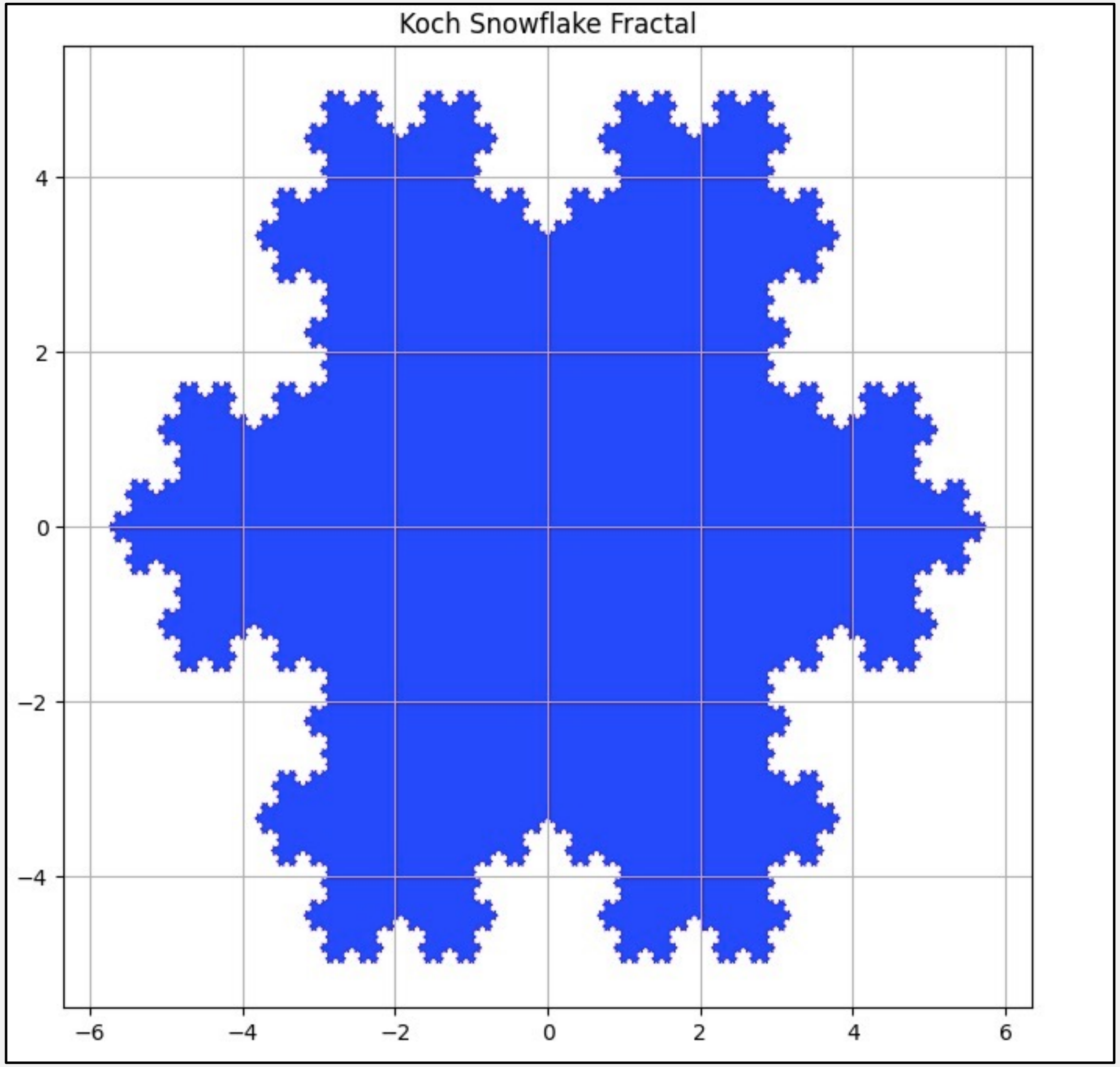
➢ The statistical properties of Brownian motion are invariant under time translation, meaning that the motion's probabilistic behavior does not change over time. The increments over disjoint time intervals are independent.

4. Scaling Property:

➢ Brownian motion has a specific scaling property: scaling down time and space by a factor of c yields another Brownian motion. (**fractal structure)**

III-THE ARITHMETIC BROWNIAN MOTION

- **Scaling Property illustration with KOCH snowflake Fractal**



IV-GEOMETRIC BROWNIAN MOTION (GBM)

IV-THE GEOMETRIC BROWNIAN MOTION

❖ Defition of the Geometric Brownian motion

- GBM is a common model used in finance to represent the stochastic evolution of continuously compounded returns of a financial instrument, such as stock prices, over time.
- In a GBM, the continuous return of the asset price is normally distributed, and the model takes into account both the drift and the randomness in the returns.

IV-THE GEOMETRIC BROWNIAN MOTION

❖ **Characteristics of Brownian Motion:**

1. **Stochastic Process:**

   ➢ **GBM is a continuous-time stochastic process**, meaning it's random and evolves over time. It’s often used in finance to model stock prices, exchange rates, and other financial derivatives.

2. **Non-Negative Values**:

   ➢ GBM ensures that the values remain non-negative, making it particularly suited for modeling quantities like stock prices that cannot fall below zero.

3. **Drift and Volatility:**

   - Drift: The trend that consistently influences the motion, typically representing the expected return of an asset.
   - Volatility: A measure of how much the asset’s price is expected to fluctuate, determining the dispersion around the drift.

IV-THE GEOMETRIC BROWNIAN MOTION

❖**Characteristics of Brownian Motion:**

4. **Log-Normal Distribution:**

➢ The end values of a GBM process are log-normally distributed: log returns follows a normal distribution.

5. **Continuous and non Differentiable Paths:**

➢ GBM paths are continuous **but not everywhere differentiable**, similar to Brownian motion. It portrays smooth paths yet exhibits randomness and unpredictability: This means that, although the path is continuous in time, you can't find a tangent to the path at almost any point—it's too "rough.

5. **Exponential Growth with Stochastic Fluctuations:**

➢ GBM is described mathematically as an exponential function multiplied by a stochastic process, leading to an exponential mean growth trend with stochastic fluctuations.

IV-THE GEOMETRIC BROWNIAN MOTION

❖**Characteristics of Brownian Motion:**

5. **Markov Property:**

➢GBM has the Markov property, meaning the future evolution of the process depends only on the current state, not on how it arrived at that state.

5.**Memorylessness:**

➢Owing to the Markov property, GBM is memoryless—the future paths are independent of the past, given the present.

5.**Used in Option Pricing:**

➢It's foundational in the Black-Scholes-Merton model for option pricing, where it's used to model the underlying asset price dynamics.

5. **Random Walk:**

➢It can be considered as a type of random walk, especially in the context of finance, where price changes are often assumed to be random and independent.

IV-THE GEOMETRIC BROWNIAN MOTION

❖**ZOOM on the Non-Differentiability and Quadratic Variation in Stochastic Processes:**

- The formula for the quadratic variation of a stochastic process, particularly for a Brownian motion BtBt, over a partition of an interval $[0, T]$ is given by
- $[\boldsymbol{B}]\boldsymbol{T} = \sum_{i=0}^{n-1}\left(\boldsymbol{Bti} + \mathbf{1} - \boldsymbol{Bti})^2\right)$ where $0 = t0 < t1 < t2 < \cdots < tn = T$ is a partition of the interval $[0, T]$
- In the context of standard Brownian motion, the quadratic variation over the interval [0,T]

$$[\boldsymbol{B}]\boldsymbol{T} = \boldsymbol{T}$$

➢Stochastic processes like Brownian motion and Geometric Brownian Motion (GBM) are characterized by non-differentiability, meaning their paths are too "rough" to have a tangent at almost any point. This "roughness" is intrinsic due to the random, unpredictable nature of these processes.

IV-THE GEOMETRIC BROWNIAN MOTION

❖ **ZOOM on the Non-Differentiability and Quadratic Variation in Stochastic Processes:**

➢ Quadratic variation comes into play as a mathematical measure of this "roughness." In the context of Brownian motion, the quadratic variation is equal to the time interval over which it's measured, indicating a linear increase of roughness with time.

➢ **While non-differentiability is a fundamental attribute of stochastic processes due to their random, volatile nature, quadratic variation serves as a tool to quantify and analyze this attribute.** It's particularly vital in finance, contributing to the understanding of options pricing, risk management, and the general behavior of financial derivatives and assets under uncertainty.

IV-THE GEOMETRIC BROWNIAN MOTION

❖ **Mathematical representation the Geometric Brownian motion**

➢ **The differential equation representing GBM is:**

$$dSt = \mu Stdt + \sigma StdWt$$

where,

St: the stock price at time t,

μ: the expected return or "drift" coefficient,

σ: the volatility or "diffusion" coefficient,

Wt: a standard Brownian motion.

IV-THE GEOMETRIC BROWNIAN MOTION

❖ **Characteristics of Brownian Motion:**

➢ The log returns don't explicitly appear in the differential equation for Geometric Brownian Motion (GBM) that I provided. However, they are implicitly involved when we consider the continuous compounding of returns.

- The equation $\boldsymbol{dSt = \mu Stdt + \sigma StdWt}$ describes the change in the stock price $\boldsymbol{St}$.
- Now, if we integrate this equation over a time interval, and apply Ito's lemma if necessary, we'd end up with a formula for the log return of the stock over that interval. Specifically, we can rewrite the equation in terms of the natural logarithm of the stock price, $\boldsymbol{d(\ln St) = (\mu - 0.5\sigma^2)dt + \sigma dWt}$ (*) $\int (1/St)\, dSt = \ln | St | + C$

IV-THE GEOMETRIC BROWNIAN MOTION

❖Characteristics of Brownian Motion:

- Converting this into its exponential form by taking the exponent of both sides, we get

$$St = S_0. \exp(\mu - 0.5\sigma^\wedge 2\ t + \sigma Wt)$$

➢Here, S_0 is the initial asset price at $t = 0$.

➢This expression gives the asset price at any time t t in an exponential form, capturing both the deterministic and stochastic influences on the asset's price evolution.

➢**The quadratic variation over time is embedded within the Brownian motion term Wt in the equation.**

V-FROM GBM TO THE BLACK AND SCHOLES MODEL

V-FROM GBM TO THE BLACK-SCHOLES MODEL

- **Step 1: Geometric Brownian Motion**

The Geometric Brownian Motion (GBM) is characterized by the stochastic differential equation

$$dSt = \mu Stdt + \sigma StdWt$$

- Step 2: Applying Ito’s Lemma

We transform the GBM into a logarithmic form using Ito's lemma. The transformed equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

- Step 3: Black-Scholes PDE

We seek a function $V(S\_t, t)$ that gives the option’s price at time t for a given asset price S_t. By applying Ito’s lemma to V and eliminating risk through hedging, we derive the Black-Scholes partial differential equation (PDE).

V-FROM GBM TO THE BLACK-SCHOLES MODEL

$$C(S,t) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2)$$

and for a European put option

$$P(S,t) = Ke^{-r(T-t)}N(d_2) - S_t N(-d_1),$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

THANK YOU!